10.11 Suppose that a stock price $S$ follows geometric Brownian motion with expected
return $\mu$ and volatility $\sigma$:

$$dS = \mu S \, dt + \sigma S \, dz$$

What is the process followed by the variable $S^n$? Show that $S^n$ also follows geo-
metric Brownian motion. The expected value of $S_T$, the stock price at time $T$, is
$S_0 e^{\mu(T-t)}$. What is the expected value of $S^n_T$?

*10.12. Suppose that $x$ is the yield to maturity with continuous compounding on a discount
bond that pays off $\$1$ at time $T$. Assume that $x$ follows the process

$$dx = a(x_0 - x) \, dt + sx \, dz$$

where $a$, $x_0$, and $s$ are positive constants and $dz$ is a Wiener process. What is the
process followed by the bond price?

*10.13. Suppose that $x$ is the yield on a perpetual government bond that pays interest at the
rate of $\$1$ per annum. Assume that $x$ is expressed with continuous compounding,
that interest is paid continuously on the bond, and that $x$ follows the process

$$dx = a(x_0 - x) \, dt + sx \, dz$$

where $a$, $x_0$, and $s$ are positive constants and $dz$ is a Wiener process. What is the
process followed by the bond price? What is the expected instantaneous return
(including interest and capital gains) to the holder of the bond?

**APPENDIX 10A: DERIVATION OF ITO'S LEMMA**

In this appendix we show how Itô's lemma can be regarded as a natural extension
of other, simpler results. Consider a continuous and differentiable function $G$ of
a variable $x$. If $\Delta x$ is a small change in $x$ and $\Delta G$ is the resulting small change in
$G$, it is well known that

$$\Delta G \approx \frac{dG}{dx} \Delta x$$

(10A.1)

In other words, $\Delta G$ is approximately equal to the rate of change of $G$ with respect
to $x$ multiplied by $\Delta x$. The error involves terms of order $\Delta x^2$. If more precision
is required, a Taylor series expansion of $\Delta G$ can be used:

$$\Delta G = \frac{dG}{dx} \Delta x + \frac{1}{2} \frac{d^2G}{dx^2} \Delta x^2 + \frac{1}{6} \frac{d^3G}{dx^3} \Delta x^3 + \cdots$$

For a continuous and differentiable function $G$ of two variables, $x$ and $y$, the
result analogous to equation (10A.1) is

$$\Delta G \approx \frac{\partial G}{\partial x} \Delta x + \frac{\partial G}{\partial y} \Delta y$$

(10A.2)

and the Taylor series expansion of $\Delta G$ is
\[ \Delta G = \frac{\partial G}{\partial x} \Delta x + \frac{\partial G}{\partial y} \Delta y + \frac{1}{2} \frac{\partial^2 G}{\partial x^2} \Delta x^2 + \frac{\partial^2 G}{\partial x \partial y} \Delta x \Delta y + \frac{1}{2} \frac{\partial^2 G}{\partial y^2} \Delta y^2 + \cdots \]  

(10A.3)

In the limit as \( \Delta x \) and \( \Delta y \) tend to zero, equation (10A.3) gives

\[ dG = \frac{\partial G}{\partial x} dx + \frac{\partial G}{\partial y} dy \]  

(10A.4)

A derivative is a function of a variable that follows a stochastic process. We now extend equation (10A.4) to cover such functions. Suppose that a variable \( x \) follows the general Ito process in equation (10.4):

\[ dx = a(x, t) dt + b(x, t) dz \]  

(10A.5)

and that \( G \) is some function of \( x \) and of time, \( t \). By analogy with equation (10A.3), we can write

\[ \Delta G = \frac{\partial G}{\partial x} \Delta x + \frac{\partial G}{\partial y} \Delta y + \frac{1}{2} \frac{\partial^2 G}{\partial x^2} \Delta x^2 + \frac{\partial^2 G}{\partial x \partial y} \Delta x \Delta y + \frac{1}{2} \frac{\partial^2 G}{\partial y^2} \Delta y^2 + \cdots \]  

(10A.6)

Equation (10A.5) can be discretized to

\[ \Delta x = a(x, t) \Delta t + b(x, t) \epsilon \sqrt{\Delta t} \]  

(10A.7)

This equation reveals an important difference between the situation in equation (10A.6) and the situation in equation (10A.3). When limiting arguments were used to move from equation (10A.3) to equation (10A.4), terms in \( \Delta x^2 \) were ignored because they were second-order terms. From equation (10A.7),

\[ \Delta x^2 = b^2 \epsilon^2 \Delta t + \text{terms of higher order in } \Delta t \]  

(10A.8)

which shows that the term involving \( \Delta x^2 \) in equation (10A.6) has a component of order \( \Delta t \) and cannot be ignored.

The variance of a standardized normal distribution is 1.0. This means that

\[ E(\epsilon^2) - [E(\epsilon)]^2 = 1 \]

where \( E \) denotes expected value. Since \( E(\epsilon) = 0 \), it follows that \( E(\epsilon^2) = 1 \). The expected value of \( \epsilon^2 \Delta t \) is therefore \( \Delta t \). It can be shown that the variance of \( \epsilon^2 \Delta t \) is of order \( \Delta t^2 \) and that as a result of this, \( \epsilon^2 \Delta t \) becomes nonstochastic and equal to its expected value of \( \Delta t \) as \( \Delta t \) tends to zero. It follows that the first term on the right-hand side of equation (10A.8) becomes nonstochastic and equal to \( b^2 \Delta t \) as \( \Delta t \) tends to zero. Taking limits as \( \Delta x \) and \( \Delta t \) tend to zero in equation (10A.6), and
using this last result, we therefore obtain

$$dG = \frac{\partial G}{\partial x} dx + \frac{\partial G}{\partial t} dt + \frac{1}{2} \frac{\partial^2 G}{\partial x^2} b^2 dt$$  \hspace{1cm} (10A.9)$$

This is Ito’s lemma. Substituting for $dx$ from equation (10A.5), equation (10A.9) becomes

$$dG = \left( \frac{\partial G}{\partial x} a + \frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial x^2} b^2 \right) dt + \frac{\partial G}{\partial x} b \, dz$$

$$\mathcal{V}(e^2 \Delta t) = \mathcal{E} \left\{ \left[ e^2 \Delta t - \mathcal{E} \left\{ e^2 \Delta t \right\} \right]^2 \right\}$$

$$= \mathcal{E} \left\{ \left[ e^2 \Delta t - e^2 \right]^2 \right\}$$

$$= \mathcal{E} \left\{ e^2 \Delta t \cdot [e^2 - 1]^2 \right\}$$

$$= \Delta t^2 \, \mathcal{E} \left\{ [e^2 - 1]^2 \right\}$$

$$= \Delta t^2 \, \mathcal{E} \left\{ e^4 - 2 e^2 + 1 \right\}$$

$$= \Delta t^2 \, \left\{ \mathcal{E} \left\{ e^4 \right\} \right\} - \Delta t^2 \, \mathcal{E} \left\{ e^2 \right\} + \Delta t^2 \, \mathcal{E} \left\{ e^2 \right\}$$

$$= 2 \Delta t^2 \left\{ \mathcal{E} \left\{ e^2 \right\} \right\}$$

$$\text{cov}(x, y) = \mathcal{E}(x \cdot y) - \mathcal{E} x \cdot \mathcal{E} y$$

1. **Ito process:** $dx = a(x, t) \, dt + b(x, t) \, dz$

2. **Taylor series with two terms for $G(x, t)$:**

   $$dG = \frac{dG}{dx} dx + \frac{1}{2} \frac{d^2G}{dx^2} dx^2 + \frac{dG}{dt} dt + \frac{1}{2} \frac{d^2G}{dt^2} dt^2$$

3. Insert Ito process into Taylor series and apply **Ito’s multiplication rule:**

   $$dx^2 = [a \, dt + b \, dz]^2 = a^2 \, dt^2 + 2 \, ab \, dt \, dz + b^2 \, dz^2 = b^2 \, dt$$

   $$dx \, dt = [a \, dt + b \, dz] \, dt = a \, dt^2 + b \, dz \, dt = 0$$

   $$dG = \frac{dG}{dx} dx + \frac{dG}{dt} dt + \frac{1}{2} \frac{d^2G}{dx^2} dx^2 + \frac{dG}{dx} \, dt \, dx + \frac{1}{2} \frac{d^2G}{dt^2} \, dt^2$$

   $$dG = \frac{dG}{dx} dx + \frac{dG}{dt} dt + \frac{1}{2} \frac{d^2G}{dx^2} \, b^2 \, dt + \frac{dG}{dx} \, b \, dz$$

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*13.11. Using risk-neutral valuation arguments, show that an option to exchange one IBM share for two Kodak shares in six months has a value that is independent of the level of interest rates.

13.12. Consider a commodity with constant volatility, \( \sigma \). Assuming that the risk-free interest rate is constant, show that in a risk-neutral world,

\[
\ln S_T \sim \phi \left[ \ln F - \frac{\sigma^2}{2} (T - t), \sigma \sqrt{T - t} \right]
\]

where \( S_T \) is the value of the commodity at time \( T \) and \( F \) is the futures price for a contract maturing at time \( T \).

*13.13. What is the formula for the price of a European call option on a foreign index when the strike price is in dollars and the index is translated into dollars at a predetermined exchange rate? What difference does it make if the index is translated into dollars at the exchange rate prevailing at the time of exercise?

**APPENDIX 13A: GENERALIZATION OF ITO'S LEMMA**

Ito's lemma as presented in Appendix 10A provides the process followed by a function of a single stochastic variable. Here we present a generalized version of Ito's lemma for the process followed by a function of several stochastic variables.

Suppose that a function, \( f \), depends on the \( n \) variables \( x_1, x_2, \ldots, x_n \) and time, \( t \). Suppose further that \( x_i \) follows an Ito process with instantaneous drift \( a_i \) and instantaneous variance \( b_i^2 \) (\( 1 \leq i \leq n \)), that is,

\[
dx_i = a_i \, dt + b_i \, dz_i
\]

(13A.1)

where \( dz_i \) is a Wiener process (\( 1 \leq i \leq n \)). Each \( a_i \) and \( b_i \) may be any function of all the \( x_i \)'s and \( t \). A Taylor series expansion of \( f \) gives

\[
\Delta f = \sum_i \frac{\partial f}{\partial x_i} \Delta x_i + \frac{\partial f}{\partial t} \Delta t + \frac{1}{2} \sum_i \sum_j \frac{\partial^2 f}{\partial x_i \partial x_j} \Delta x_i \Delta x_j + \sum_j \frac{\partial^2 f}{\partial x_i \partial t} \Delta x_i \Delta t + \cdots
\]

(13A.2)

Equation (13A.1) can be discretized as

\[
\Delta x_i = a_i \Delta t + b_i \epsilon_i \sqrt{\Delta t}
\]

where \( \epsilon_i \) is a random sample from a standardized normal distribution. The correlation, \( \rho_{ij} \), between \( dz_i \) and \( dz_j \) is defined as the correlation between \( \epsilon_i \) and \( \epsilon_j \). In Appendix 10A it was argued that

\[
\lim_{\Delta t \to 0} \frac{\Delta x_i^2}{\Delta t} = b_i^2 dt
\]
Similarly, 

\[ \lim_{\Delta t \to 0} \Delta x_i \Delta x_j = b_i b_j \rho_{ij} \, dt \]

As \( \Delta t \to 0 \), the first three terms in the expansion of \( \Delta f \) in equation (13A.2) are of order \( \Delta t \). All other terms are of higher order. Hence

\[
    df = \sum_i \frac{\partial f}{\partial x_i} dx_i + \frac{\partial f}{\partial t} dt + \frac{1}{2} \sum_i \sum_j \frac{\partial^2 f}{\partial x_i \partial x_j} b_i b_j \rho_{ij} \, dt
\]

This is the generalized version of Ito’s lemma. Substituting for \( dx_i \) from equation (13A.1) gives

\[
    df = \left( \sum_i \frac{\partial f}{\partial x_i} a_i + \frac{\partial f}{\partial t} + \frac{1}{2} \sum_i \sum_j \frac{\partial^2 f}{\partial x_i \partial x_j} b_i b_j \rho_{ij} \right) dt + \sum_i \frac{\partial f}{\partial x_i} b_i dz_i
\]

(13A.3)

**APPENDIX 13B: DERIVATION OF THE GENERAL DIFFERENTIAL EQUATION SATISFIED BY DERIVATIVES**

Consider a certain derivative security that depends on \( n \) state variables and time, \( t \). We make the assumption that there are a total of at least \( n + 1 \) traded securities (including the one under consideration) whose prices depend on some or all of the \( n \) state variables. In practice, this is not unduly restrictive. The traded securities may be options with different strike prices and exercise dates, forward contracts, futures contracts, bonds, stocks, and so on. We assume that no dividends or other income is paid by the \( n + 1 \) traded securities.\(^{10}\) Other assumptions are similar to those made in Section 11.4 to derive the Black–Scholes equation.

The \( n \) state variables are assumed to follow continuous-time Ito diffusion processes. We denote the \( i \)th state variable by \( \theta_i \) (1 \( \leq i \leq n \)) and suppose that

\[
    d\theta_i = m_i \theta_i \, dt + s_i \theta_i \, dz_i
\]

(13B.1)

where \( dz_i \) is a Wiener process and the parameters, \( m_i \) and \( s_i \), are the expected growth rate in \( \theta_i \) and the volatility of \( \theta_i \). The \( m_i \) and \( s_i \) can be functions of any of the \( n \) state variables and time. Other notation used is as follows:

- \( \rho_{ik} \): correlation between \( dz_i \) and \( dz_k \) (1 \( \leq i, k \leq n \))
- \( f_j \): price of the \( j \)th traded security (1 \( \leq j \leq n + 1 \))
- \( r \): instantaneous (i.e., very short-term) risk-free rate

\(^{10}\)This is not restrictive. A non-dividend-paying security can always be obtained from a dividend-paying security by reinvesting the dividends in the security.
One of the \( f_j \) is the price of the security under consideration. The short-term risk-free rate, \( r \), may be one of the \( n \) state variables.

Since the \( n + 1 \) traded securities are all dependent on the \( \theta_i \), it follows from Ito’s lemma in Appendix 13A that the \( f_j \) follow diffusion processes:

\[
df_j = \mu_j f_j \, dt + \sum_{i=1}^{n} \sigma_{ij} f_j \, dz_i
\]  

(13B.2)

where

\[
\mu_j f_j = \frac{\partial f_j}{\partial t} + \sum_{i} \frac{\partial f_j}{\partial \theta_i} m_i \theta_i + \frac{1}{2} \sum_{i,k} \rho_{ik} s_i \theta_i s_k \theta_k \frac{\partial^2 f_j}{\partial \theta_i \partial \theta_k}
\]  

(13B.3)

and

\[
\sigma_{ij} f_j = \frac{\partial f_j}{\partial \theta_i} s_j \theta_i
\]  

(13B.4)

In these equations, \( \mu_j \) is the instantaneous mean rate of return provided by \( f_j \) and \( \sigma_{ij} \) is the component of the instantaneous standard deviation of the rate of return provided by \( f_j \), which may be attributed to the \( \theta_i \).

Since there are \( n + 1 \) traded securities and \( n \) Wiener processes in equation (13B.2), it is possible to form an instantaneously riskless portfolio, \( \Pi \), using the securities. Define \( k_j \) as the amount of the \( j \)th security in the portfolio, so that

\[
\Pi = \sum_{j=1}^{n+1} k_j f_j
\]  

(13B.5)

The \( k_j \) must be chosen so that the stochastic components of the returns from the securities are eliminated. From equation (13B.2) this means that

\[
\sum_{j=1}^{n+1} k_j \sigma_{ij} f_j = 0 \quad \text{for } 1 \leq i \leq n
\]  

(13B.6)

The return from the portfolio is then given by

\[
d\Pi = \sum_{j=1}^{n+1} k_j \mu_j f_j \, dt
\]

The cost of setting up the portfolio is \( \sum_{j} k_j f_j \). If there are no arbitrage opportunities, the portfolio must earn the risk-free interest rate, so that

\[
\sum_{j=1}^{n+1} k_j \mu_j f_j = \pi t r
\]

(13B.7)

or

\[
\sum_{j=1}^{n+1} k_j f_j (\mu_j - r) = 0
\]  

(13B.8)
Equations (13B.6) and (13B.8) can be regarded as \( n + 1 \) homogeneous linear equations in the \( k_j \)'s. The \( k_j \)'s are not all zero. From a well-known theorem in linear algebra, equations (13B.6) and (13B.8) can be consistent only if

\[
f_j(\mu - r) = \sum_{i=1}^{n} \lambda_i \sigma_{ij} f_j	ag{13B.9}
\]

or

\[
\mu - r = \sum_{i=1}^{n} \lambda_i \sigma_{ij}
\]

for some \( \lambda_i \) \((1 \leq i \leq n)\), which are dependent only on the state variables and time. This proves the result in equation (13.13).

Substituting from equations (13B.3) and (13B.4) into equation (13B.9), we obtain

\[
\frac{\partial f_j}{\partial t} + \sum_i \theta_i \frac{\partial f_j}{\partial \theta_i} m_i - \lambda_i s_i + \frac{1}{2} \sum_{i,k} \rho_{ik} \theta_i \theta_k \frac{\partial^2 f_j}{\partial \theta_i \partial \theta_k} - r f_j = \sum_i \lambda_i \frac{\partial f_j}{\partial \theta_i} s_i \theta_i
\]

which reduces to

\[
\frac{\partial f_j}{\partial t} + \sum_i \theta_i \frac{\partial f_j}{\partial \theta_i} (m_i - \lambda_i s_i) + \frac{1}{2} \sum_{i,k} \rho_{ik} \theta_i \theta_k \frac{\partial^2 f_j}{\partial \theta_i \partial \theta_k} = r f_j
\]

Dropping the subscripts to \( f \), we deduce that any security whose price, \( f \), is contingent on the state variables \( \theta_i \)(\( 1 \leq i \leq n \)) and time, \( t \), satisfies the second-order differential equation

\[
\frac{\partial f}{\partial t} + \sum_i \theta_i \frac{\partial f}{\partial \theta_i} (m_i - \lambda_i s_i) + \frac{1}{2} \sum_{i,k} \rho_{ik} \theta_i \theta_k \frac{\partial^2 f}{\partial \theta_i \partial \theta_k} = r f
\]

(13B.11)

The particular derivative security that is obtained is determined by the boundary conditions that are imposed on equation (13B.11).